

Free product von Neumann algebras associated to graphs, and Guionnet, Jones, Shlyakhtenko subfactors in infinite depth

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Abstract

Given a subfactor planar algebra \mathcal{P} , Guionnet, Jones and Shlyakhtenko give a diagrammatic construction of a II_1 subfactor whose planar algebra is \mathcal{P} . They showed if \mathcal{P} is finite-depth, then the factors are interpolated free group factors, and they identified the parameters. We prove if \mathcal{P} is infinite-depth, then the factors are isomorphic to $L(\mathbb{F}_\infty)$.

1 Introduction

In [Jon83], Jones initiated the study of modern subfactor theory. Given a finite index II_1 subfactor $A_0 \subset A_1$, one computes its standard invariant: two towers $(A'_0 \cap A_j : j \geq 0)$ and $(A'_1 \cap A_j : j \geq 1)$ of finite dimensional von Neumann algebras [Jon83]. The standard invariant has been axiomatized by Ocneanu's paragroups [Ocn88], Popa's λ -lattices [Pop95], and Jones' subfactor planar algebras [Jon10]. Popa showed that given a standard invariant \mathcal{P} , we can reconstruct a II_1 subfactor $A_0 \subset A_1$ whose standard invariant is \mathcal{P} [Pop95]. Guionnet, Jones, and Shlyakhtenko [GJS10] give a planar-algebraic proof of the above result. Moreover, if \mathcal{P} is finite depth with loop parameter $\delta > 1$, they showed that A_k , the k^{th} factor in the Jones tower, is isomorphic to $L(\mathbb{F}(1 + 2\delta^{-2k}(\delta - 1)I))$ where I is the global index of \mathcal{P} [GJS11]. Kodiyalam and Sunder also obtained this formula when \mathcal{P} is depth 2 [KS09a, KS09b]. In this paper, we prove the following theorem:

Theorem. *If \mathcal{P} is infinite depth, then every factor in the construction of [GJS10] is isomorphic to $L(\mathbb{F}_\infty)$.*

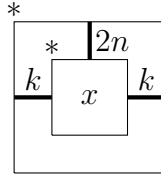
Using this theorem, we recover a diagrammatic proof of a result of Popa and Shlyakhtenko for \mathcal{P} infinite depth [PS03]:

Corollary. *Every infinite depth subfactor planar algebra is the standard invariant of $\mathcal{N} \subset \mathcal{M}$ where $\mathcal{N}, \mathcal{M} \cong L(\mathbb{F}_\infty)$.*

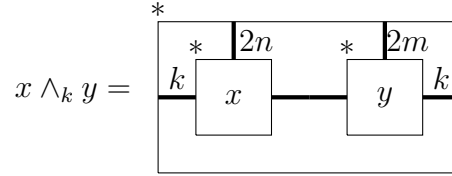
1.1 Outline of the proof

To prove the above theorem, we will bootstrap the proof from [GJS11] of the finite-depth case to the infinite-depth case using standard embedding tricks.

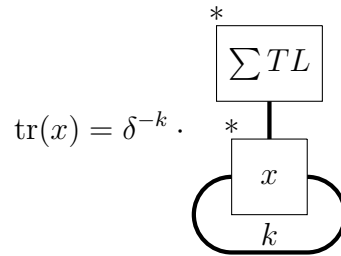
The GJS Construction: We will recall the construction of Guionnet, Jones and Shlyakhtenko. For more details, see [GJS10] and [GJS11]. Let $\mathcal{P} = (\mathcal{P}_n^\pm)_{n \geq 0}$ be a subfactor planar algebra with loop parameter $\delta > 1$ (see [Jon10] for the definition of a subfactor planar algebra). Set $Gr_k(\mathcal{P}^+) = \bigoplus_{n \geq 0} \mathcal{P}_{k,n}^+$ where $\mathcal{P}_{k,n}^+ = \mathcal{P}_{k+n}^+$ and an element of $\mathcal{P}_{k,n}^+$ is represented as



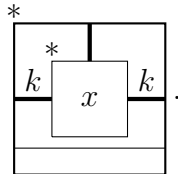
where the $*$ is always in an unshaded region and a thick string with a j next to it denotes j parallel strings. If $x \in \mathcal{P}_{n+k}^+$ and $y \in \mathcal{P}_{m+k}^+$ then define a multiplication \wedge_k by



which is an element in $\mathcal{P}_{k,m+n}^+$. One can endow $Gr_k(\mathcal{P}^+)$ with the following trace: if $x \in \mathcal{P}_{k,n}^+$ then



where $\sum TL$ denotes the sum of all Temperley-Lieb diagrams, i.e. all planar pairings of the $2n$ strings on top of x . This trace is positive definite, and one can form the von Neumann algebra A_k which is the strong closure of $Gr_k(\mathcal{P}^+)$ acting on $L^2(Gr_k(\mathcal{P}^+))$ by left multiplication (under \wedge_k). It is shown that A_k is a II_1 factor. Moreover one can view $x \in A_k$ as an element in A_{k+1} as follows:

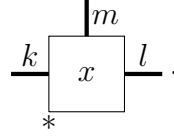


With this identification, A_k is a von Neumann subalgebra of A_{k+1} and $A_0 \subset A_1 \subset \dots \subset A_k \subset \dots$ is a Jones tower of II_1 factors with standard invariant \mathcal{P}^+ .

To identify the isomorphism type of the A_k , we look at the semi-finite algebra

$$V_+ = \bigoplus_{k+l+m \text{ even}} \mathcal{P}_{k,l,m}^+$$

where $\mathcal{P}_{k,l,m}^+ = \mathcal{P}_{\frac{k+l+m}{2}}^+$ and is spanned by boxes of the form



The element x above is identified with the following element of $\mathcal{P}_{k+2p,l+2q,m}^+$:

$$\delta^{-(p+q)/2} \begin{array}{c} \text{Diagram: A box labeled } x \text{ with a vertical line labeled } m \text{ on top, a horizontal line labeled } k \text{ on the left, and a horizontal line labeled } l \text{ on the right. The box is enclosed in a larger rectangle. On the left side of the rectangle, there are } p \text{ cups (circles) labeled } p. On the right side, there are } q \text{ cups labeled } q. A small asterisk is at the bottom left corner.} \end{array} \quad (1)$$

where there are p cups on the left and q cups on the right. Under these identifications, V_+ completes to a semifinite von Neumann algebra, \mathcal{M}_+ where the multiplication is given by

$$\left(\begin{array}{c} \text{Diagram: Box } x \text{ with top line } m, \text{ left line } k, \text{ right line } l. \text{ Asterisk at bottom left.} \end{array} \right) \cdot \left(\begin{array}{c} \text{Diagram: Box } y \text{ with top line } m', \text{ left line } k', \text{ right line } l'. \text{ Asterisk at bottom left.} \end{array} \right) = \delta_{l,k'} \begin{array}{c} \text{Diagram: Two boxes } x \text{ and } y \text{ side-by-side. Box } x \text{ has top line } n, \text{ left line } k, \text{ right line } l. \text{ Box } y \text{ has top line } m', \text{ left line } l, \text{ right line } l'. \text{ Both have asterisks at bottom left.} \end{array}$$

where we have assumed that we have added enough cups as in diagram (1) so that l and k' are either the same or differ by 1. The trace on \mathcal{M}_+ is given by

$$\text{Tr}(x) = \begin{array}{c} \text{Diagram: A box labeled } x \text{ with a vertical line labeled } k \text{ on the left and a vertical line labeled } k \text{ on the right, forming a loop. A small asterisk is at the bottom left corner. Above the box is a box labeled } \sum TL \text{ with a vertical line connecting it to the top of the } x \text{ box.} \end{array}$$

provided that the number of strings on the left and right of x have the same parity and is zero otherwise. It is easy to check that the identification in diagram (1) respects both the trace and multiplication.

The algebras A_{2k} above are a compression of \mathcal{M}_+ by the projection p_{2k}^+ where for general n ,

$$p_n^+ = \begin{array}{|c|} \hline n \\ \hline \\ \hline \end{array}^*$$

Similarly, we can consider a semifinite von Neumann algebra \mathcal{M}_- generated by the \mathcal{P}_n^- 's (where the $*$ is in a *shaded* region and on the bottom of the box), and if we define projections p_n^- , then A_{2k+1} is the compression of \mathcal{M}_- by p_{2k+1}^- .

A diagrammatic argument shows that \mathcal{M}_+ is generated by

$$\mathcal{A}_+ = \left(\bigcup_{k,\ell} \mathcal{P}_{k,\ell,0}^+ \right)'' \quad \text{and} \quad X = s - \lim_{k \rightarrow \infty} \begin{array}{|c|} \hline \text{---} \\ \hline 2k \\ \hline \end{array}^* + \begin{array}{|c|} \hline \text{---} \\ \hline 2k \\ \hline \end{array}^*$$

where the limit above is in the strong operator topology. This element is an \mathcal{A}_+ -valued semicircular element in the sense of [Shl99] and is used in the calculation of the isomorphism class of the algebras A_k

The finite depth case: Let Γ denote the principal graph of \mathcal{P} with edge set $E(\Gamma)$ and initial vertex $*$. Let $\ell^\infty(\Gamma)$ as the von Neumann algebra of bounded functions on the vertices of Γ and endow $\ell^\infty(\Gamma)$ with a trace tr such that $\text{tr}(p_v) = \mu_v$, where p_v is the delta function at v and μ_v is the entry corresponding to a fixed Perron-Frobenius eigenvector for Γ with $\mu_* = 1$. From [GJS11], $A_0 = p_* \mathcal{M}(\Gamma) p_*$ where $\mathcal{M}(\Gamma)$ is the von Neumann algebra generated by $(\ell^\infty(\Gamma), \text{tr})$ and $\ell^\infty(\Gamma)$ -valued semicircular elements $\{X_e : e \in E(\Gamma)\}$ which are compressions of X by partial isometries in \mathcal{A}_+ and are free with amalgamation over $\ell^\infty(\Gamma)$. Each X_e is supported under $p_v + p_w$, where e connects v and w , and we have $X_e = p_v X_e p_w + p_w X_e p_v$. Assuming that $\mu_v \geq \mu_w$, the scalar-valued distribution of X_e^2 in $(p_v + p_w) \mathcal{M}(\Gamma) (p_v + p_w)$ is free-Poisson with an atom of size $\frac{\mu_v - \mu_w}{\mu_v + \mu_w}$ at 0. Therefore,

$$vN(\ell^\infty(\Gamma), X_e) = L(\mathbb{Z}) \otimes M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \ell^\infty(\Gamma \setminus \{v, w\})$$

with $p_w = (1 \otimes e_{1,1}) \oplus 0 \oplus 0$ and $p_v = (1 \otimes e_{2,2}) \oplus 1 \oplus 0$, where $\{e_{i,j} : 1 \leq i, j \leq 2\}$ is a system of matrix units for $M_2(\mathbb{C})$. If Γ is finite, Dykema's formulas for computing certain amalgamated free products [Dyk93, Dyk11] show that $\mathcal{M}(\Gamma)$ is an interpolated free group factor and the compression formula gives the result for A_0 . Since A_{2n} is a δ^{2n} -amplification of A_0 , the result holds for A_{2n} . The factor A_1 is a compression of $\mathcal{M}(\Gamma^*)$ with Γ^* the dual principal graph of \mathcal{P} . Applying the same analysis to Γ^* gives the formula for the A_{2n+1} 's.

The infinite depth case: We similarly define $\mathcal{M}(\Gamma)$ for an arbitrary connected, loopless (not necessarily bipartite) graph Γ . If Γ is finite, we show that $\mathcal{M}(\Gamma) \cong L(\mathbb{F}_t) \oplus A$ where A is finite-dimensional and abelian (A can possibly be $\{0\}$). Furthermore, if p_Γ is the identity of $L(\mathbb{F}_t)$ and Γ' is a finite graph containing Γ , then

the inclusion $p_\Gamma \mathcal{M}(\Gamma) p_\Gamma \rightarrow p_\Gamma \mathcal{M}(\Gamma') p_\Gamma$ is a standard embedding of interpolated free group factors (see Definition 2.2 and Remark 2.3 below). Therefore, if \mathcal{P} is infinite depth with principal graph Γ , we write Γ as an increasing union of finite graphs Γ_k where Γ_k is Γ truncated at depth k . Since standard embeddings are preserved by cut-downs, the inclusion $p_* \mathcal{M}(\Gamma_k) p_* \rightarrow p_* \mathcal{M}(\Gamma_{k+1}) p_*$ is a standard embedding. As A_0 is the inductive limit of the $p_* \mathcal{M}(\Gamma_k) p_*$'s, it is an interpolated free group factor where the parameter is the limit of the parameters for the $p_* \mathcal{M}(\Gamma_k) p_*$'s, which is ∞ . Since the factors A_{2k} are amplifications of A_0 , $A_{2k} \cong L(\mathbb{F}_\infty)$. Applying the same analysis to Γ^* (the dual principal graph of \mathcal{P}) shows that $A_{2k+1} \cong L(\mathbb{F}_\infty)$.

Organization: Section 2 covers some preliminary material on interpolated free group factors, free dimension, and standard embeddings. Section 3 introduces $\mathcal{M}(\Gamma)$ and establishes both its structure and how it includes into $\mathcal{M}(\Gamma')$ for Γ a subgraph of Γ' . Section 4 provides the proof that the factors A_k above are all isomorphic to $L(\mathbb{F}_\infty)$.

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2 Preliminaries

Dykema [Dyk94] and Rădulescu [Răd94] independently developed interpolated free group factors $L(\mathbb{F}_t)$ for $1 < t \leq \infty$. These coincide with the usual free group factors when $t \in \mathbb{N} \cup \{\infty\}$ and they satisfy

$$L(\mathbb{F}_t) * L(\mathbb{F}_s) = L(\mathbb{F}_{s+t}) \text{ and } L(\mathbb{F}_t)_\gamma = L(\mathbb{F}(1 + \gamma^{-2}(t-1))),$$

where M_γ is the γ -amplification of the II_1 factor M . It is known that either the interpolated free group factors are all isomorphic or they are pairwise non-isomorphic [Dyk94, Răd94].

Notation 2.1. Throughout this paper, we will be concerned with finite von Neumann algebras (\mathcal{M}, tr) which can be written in the form

$$\mathcal{M} = \bigoplus_{\gamma_0}^{p_0} \mathcal{M}_0 \oplus \bigoplus_{j \in J}^{p_j} L(\mathbb{F}_{t_j}) \oplus \bigoplus_{k \in K}^{q_k} M_{n_k}$$

where \mathcal{M}_0 is a diffuse hyperfinite von Neumann algebra, $L(\mathbb{F}_{t_j})$ is an interpolated free group factor with parameter t_j , M_{n_k} is the algebra of $n_k \times n_k$ matrices over the scalars, and the sets J and K are at most finite and countably infinite respectively. We use p_j to denote the projection in $L(\mathbb{F}_{t_j})$ corresponding to the identity of $L(\mathbb{F}_{t_j})$ and q_k to denote a minimal projection in M_{n_k} . The projections p_j and q_k have traces γ_j and α_k respectively. Let p_0 be the identity in \mathcal{M}_0 with trace γ_0 . We write $\overset{p,q}{M}_2$ to mean M_2 with a choice of minimal orthogonal projections p and q .

If the interpolated free group factors turn out to be non-isomorphic, it is desirable to be able to calculate which interpolated free group factors appear in amalgamated free products. To help facilitate this calculation, Dykema defined the notion of free dimension. In general, one has

$$\text{fdim}(\mathcal{M}_1 *_D \mathcal{M}_2) = \text{fdim}(\mathcal{M}_1) + \text{fdim}(\mathcal{M}_2) - \text{fdim}(D)$$

whenever \mathcal{M}_1 and \mathcal{M}_2 are of the form of Notation 2.1 and D is finite dimensional [Dyk93, Dyk95, Dyk11, DR11]. In general, for the algebra \mathcal{M} in Notation 2.1, we have

$$\text{fdim}(\mathcal{M}) = 1 + \sum_{j \in J} \gamma_j^2(t_j - 1) - \sum_{k \in K} \alpha_k^2.$$

Notice that this includes the special case $\text{fdim}(L(\mathbb{F}_t)) = t$.

Of course if the interpolated free group factors are isomorphic, then the free dimension is not well defined; however, the only purpose of the free dimension is to determine the parameter of interpolated free group factors which show up in amalgamated free products. Therefore all the lemmas below will remain valid if all references to free dimension are removed.

Of critical importance will be the notion of a *standard embedding* of interpolated free group factors [Dyk93]. This is a generalization of a mapping $\mathbb{F}_n \rightarrow \mathbb{F}_m$ for $m > n$ sending the n generators of \mathbb{F}_n onto n of the m generators for \mathbb{F}_m .

Definition 2.2. Let $1 < r < s$ and $\phi : L(\mathbb{F}_r) \rightarrow L(\mathbb{F}_s)$ be a von Neumann algebra homomorphism. We say that ϕ is a standard embedding if there exist nonempty sets $S \subset S'$, a family of projections $\{p_{s'} : s' \in S'\}$ with $p_{s'} \in R$ (the hyperfinite II_1 factor), a free family $\{X^{s'} : s' \in S'\}$ of semicircular elements which are also free from R , and isomorphisms

$$\alpha : L(\mathbb{F}_r) \rightarrow (R \cup \{p_s X^s p_s\}_{s \in S})'' \text{ and } \beta : L(\mathbb{F}_s) \rightarrow (R \cup \{p_{s'} X^{s'} p_{s'}\}_{s' \in S'})''$$

such that $\phi = \beta^{-1} \circ \iota \circ \alpha$ where $\iota : (R \cup \{p_s X^s p_s\}_{s \in S})'' \rightarrow (R \cup \{p_{s'} X^{s'} p_{s'}\}_{s' \in S'})''$ is the canonical inclusion. We will write $A \xrightarrow{s.e.} B$ to mean that the inclusion of A into B is a standard embedding.

Remark 2.3. Dykema in [Dyk93] and [Dyk95] shows the following useful properties of standard embeddings which we will use extensively in this paper.

- (1) If A is an interpolated free group factor, the canonical inclusion $A \rightarrow A * \mathcal{M}$ is a standard embedding whenever \mathcal{M} is of the form in Notation 2.1.
- (2) A composite of standard embeddings is a standard embedding.
- (3) If $A_n = L(\mathbb{F}_{s_n})$ with $s_n < s_{n+1}$ for all n and $\phi_n : A_n \xrightarrow{s.e.} A_{n+1}$, then the inductive limit of the A_n with respect to the ϕ_n is $L(\mathbb{F}_s)$ where $s = \lim_{n \rightarrow \infty} s_n$.
- (4) If $t > s$ then $\phi : L(\mathbb{F}_s) \xrightarrow{s.e.} L(\mathbb{F}_t)$ if and only if for any nonzero projection $p \in L(\mathbb{F}_s)$, $\phi|_{pL(\mathbb{F}_s)p} : pL(\mathbb{F}_s)p \xrightarrow{s.e.} \phi(p)L(\mathbb{F}_t)\phi(p)$.

Our work will rely heavily on the following two lemmas.

Lemma 2.4 ([DR11]). *Let $\mathcal{N} = (M_n(\mathbb{C}) \oplus B) \underset{D}{*}^p C$ and $\mathcal{M} = (M_n(\mathbb{C}) \otimes A \oplus B) \underset{D}{*} C$ where A , B and C are finite von Neumann algebras and D is a finite dimensional abelian von Neumann algebra. Let E be the trace-preserving conditional expectation of \mathcal{M} onto D . Assume p lies under a minimal projection in D and $E|_{M_n(\mathbb{C}) \otimes A} = E|_{M_n(\mathbb{C})} \otimes \text{tr}_A$. Then $p\mathcal{M}p = p\mathcal{N}p * A$ and the central support of p in \mathcal{M} is the same as that in \mathcal{N} .*

Lemma 2.5 ([DR11]). *Let $\mathcal{N} = (M_m(\mathbb{C}) \underset{\gamma}{\oplus}^p M_{n-m}(\mathbb{C}) \underset{\gamma}{\oplus}^q B) \underset{D}{*} C$ and $\mathcal{M} = (M_n(\mathbb{C}) \underset{\gamma}{\oplus} B) \underset{D}{*} C$ with B , C , D as in Lemma 2.4. Assume p and q sit under minimal projections in D and p is equivalent to q in \mathcal{N} . Then $p\mathcal{M}p = p\mathcal{N}p * L(\mathbb{Z})$ and the central support of p in \mathcal{M} is the same as that in \mathcal{N} .*

Note that if A , B and C are in the form in Notation 2.1, and if \mathcal{N} is an interpolated free group factor, then the proofs of the above lemmas in [DR11] show that $p\mathcal{N}p \rightarrow p\mathcal{M}p$ of Lemmas 2.4 and 2.5 are standard embeddings. This implies $\mathcal{N} \xrightarrow{s.e.} \mathcal{M}$ by Remark 2.3.

3 A von Neumann algebra associated to a finite connected graph

Let Γ be a connected, loopless finite graph with edge set $E(\Gamma)$ and vertex set $V(\Gamma)$. Assume further that each vertex $v \in V(\Gamma)$ is weighted by a real constant $\gamma_v > 0$ with $\sum_{v \in \Gamma} \gamma_v = 1$ (the weighting does *not* have to be the Perron-Frobenius weighting). Consider the abelian von Neumann algebra $\ell^\infty(\Gamma)$. Let p_v be the indicator function on the vertex v and its trace will be γ_v . We construct a finite von Neumann algebra associated to Γ (also see [KS11]).

Definition 3.1. Let Γ be as above, e be an edge in Γ connecting the vertices v and w , and assume $\gamma_v \geq \gamma_w$. Define

$$\mathcal{A}_e = M_2(\mathbb{C}) \underset{2\gamma_w}{\otimes} L(\mathbb{Z}) \underset{\gamma_v - \gamma_w}{\oplus}^{\frac{p_v}{\gamma_v}} \mathbb{C} \oplus \ell^\infty(\Gamma \setminus \{v, w\})$$

where the trace on $M_2 \otimes L(\mathbb{Z})$ is $\text{tr}_{M_2} \otimes \text{tr}_{L(\mathbb{Z})}$. \mathcal{A}_e includes $\ell^\infty(\Gamma)$ by setting

$$\begin{aligned} p_w &= 1 \otimes e_{1,1} \oplus 0 \oplus 0 \text{ and} \\ p_v &= 1 \otimes e_{2,2} \oplus 1 \oplus 0. \end{aligned}$$

Therefore, the trace preserving conditional expectation $E_e : \mathcal{A}_e \rightarrow \ell^\infty(\Gamma)$ has the property $E_e|_{M_2 \otimes L(\mathbb{Z})} = E_e|_{M_2} \otimes \text{tr}|_{L(\mathbb{Z})}$. We define $\mathcal{M}(\Gamma)$, the von Neumann algebra associated to Γ , by

$$\mathcal{M}(\Gamma) = \underset{\ell^\infty(\Gamma)}{*} (\mathcal{A}_e, E_e)_{e \in E(\Gamma)}.$$

Remark 3.2. If Γ is an infinite graph with a weighting that is not ℓ^1 , then we can still define $M(\Gamma)$ as in 3.1 although it will be a semifinite algebra. Given $e \in E(\Gamma)$ connecting vertices v and w , the compressed algebra $(p_v + p_w)\mathcal{A}_e(p_v + p_w)$ is still finite, and if $E_e : \mathcal{A}_e \rightarrow \ell^\infty(\Gamma)$ is the (tracial-weight) preserving conditional expectation, then E_e is clearly normal on $(p_v + p_w)\mathcal{A}_e(p_v + p_w)$ and is the identity on $(1 - p_v - p_w)\mathcal{A}_e(1 - p_v - p_w)$. Therefore one can take the algebraic free product Q of $(\mathcal{A}_e)_{e \in E(\Gamma)}$ with amalgamation over $\ell^\infty(\Gamma)$ and represent it on $L^2(Q, \text{Tr} \circ \underset{\ell^\infty(\Gamma)}{*} E_e)$ to obtain $\mathcal{M}(\Gamma)$.

Definition 3.3. Let $v, w \in V(\Gamma)$. We write $v \sim w$ if v and w are connected by at least 1 edge in Γ and denote $n_{v,w}$ be the number of edges joining v and w . We set $\alpha_v^\Gamma = \sum_{w \sim v} n_{v,w} \gamma_w$, and define $B(\Gamma) = \{v \in V(\Gamma) : \gamma_v > \alpha_v^\Gamma\}$.

For the rest of this section, we assume Γ is finite. We show that $\mathcal{M}(\Gamma)$ is the direct sum of an interpolated free group factor and a finite dimensional abelian algebra. More precisely, we prove the following theorem:

Theorem 3.4. *Let Γ and Γ' be connected, finite, loopless, and weighted graphs with at least 2 edges. Then $\mathcal{M}(\Gamma) \cong L(\mathbb{F}_{t_\Gamma})^{\oplus p^\Gamma} \oplus \bigoplus_{v \in B(\Gamma)} \mathbb{C}^{r_v^\Gamma}$ where $r_v^\Gamma \leq p_v$ and t_Γ is such that this algebra has the appropriate free dimension. Furthermore, if Γ is a subgraph of Γ' , then $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$.*

Notice that since we are assuming that all vertices have positive weight, it follows that $p_v p^\Gamma \neq 0$ for all $v \in \Gamma$. We prove Theorem 3.4 in a series of lemmas.

Lemma 3.5. *Let Γ be a finite, connected, weighted, loopless graph with 2 edges. Then $M(\Gamma)$ is of the form in Theorem 3.4.*

Proof. Set $D = \ell^\infty(\Gamma)$. There are two overlying cases to consider. One where Γ has 2 vertices and the other where Γ has 3 vertices.

Case 1: Assume that Γ has 2 vertices v, w and 2 edges e_1 and e_2 connecting v and w and without loss of generality assume $\gamma_v \geq \gamma_w$. We obtain the desired formula for $\mathcal{M}(\Gamma)$ by examining the following sequence of inclusions:

$$\begin{aligned}
\mathcal{N}_0 &= \begin{pmatrix} p_w & p_v^I & \\ \mathbb{C} \oplus & \mathbb{C} \oplus & \mathbb{C} \\ \gamma_w & \gamma_w & \gamma_v - \gamma_w \end{pmatrix} *_D \begin{pmatrix} p_w & q_v^I & \\ \mathbb{C} \oplus & \mathbb{C} \oplus & \mathbb{C} \\ \gamma_w & \gamma_w & \gamma_v - \gamma_w \end{pmatrix} \\
&\cap \\
\mathcal{N}_1 &= \begin{pmatrix} p_w, p_v^I & & \\ M_2 \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} *_D \begin{pmatrix} p_w & q_v^I & \\ \mathbb{C} \oplus & \mathbb{C} \oplus & \mathbb{C} \\ \gamma_w & \gamma_w & \gamma_v - \gamma_w \end{pmatrix} \\
&\cap \\
\mathcal{N}_2 &= \begin{pmatrix} p_w, p_v^I & & \\ M_2 \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} *_D \begin{pmatrix} p_w, q_v^I & & \\ M_2 \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} \\
&\cap \\
\mathcal{N}_3 &= \begin{pmatrix} M_2 \otimes L(\mathbb{Z}) \oplus & \mathbb{C} & \\ 2\gamma_w & \gamma_v - \gamma_w & \end{pmatrix} *_D \begin{pmatrix} p_w & & \\ M_2 \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} \\
&\cap \\
\mathcal{M}(\Gamma) &= \begin{pmatrix} M_2 \otimes L(\mathbb{Z}) \oplus & \mathbb{C} & \\ 2\gamma_w & \gamma_v - \gamma_w & \end{pmatrix} *_D \begin{pmatrix} M_2 \otimes L(\mathbb{Z}) \oplus & \mathbb{C} & \\ 2\gamma_w & \gamma_v - \gamma_w & \end{pmatrix},
\end{aligned}$$

where p_v decomposes as $(1 \otimes e_{2,2}) \oplus 1$ in \mathcal{A}_{e_1} and \mathcal{A}_{e_2} with $p_v^I = 1 \otimes e_{2,2}$ in \mathcal{A}_{e_1} and $q_v^I = 1 \otimes e_{2,2}$ in \mathcal{A}_{e_2} . From Lemma 2.4 and [Dyk93], we see that

$$p_v \mathcal{N}_0 p_v = \begin{pmatrix} p_v^I & & \\ \mathbb{C} \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} *_D \begin{pmatrix} q_v^I & & \\ \mathbb{C} \oplus & \mathbb{C} & \\ \gamma_w & \gamma_v - \gamma_w & \end{pmatrix} = \begin{cases} M_2 \otimes L(\mathbb{Z}) \oplus \begin{matrix} p_v^I \wedge q_v^I \\ \mathbb{C} \\ 2\gamma_w - \gamma_v \\ \gamma_v \end{matrix} & \text{if } 2\gamma_w \geq \gamma_v \\ M_2 \otimes L(\mathbb{Z}) \oplus \begin{matrix} (p_v - p_v^I) \wedge (p_v - q_v^I) \\ \mathbb{C} \\ 2\gamma_w - \gamma_v \\ \gamma_v \end{matrix} & \text{if } \gamma_v > 2\gamma_w \end{cases}$$

where in the first algebra, the identity element copy of \mathbb{C} is $p_v^I \wedge q_v^I$ and in the second algebra, the identity of the copy of \mathbb{C} is orthogonal to both p_v^I and q_v^I .

Case 1a: Assume $2\gamma_w \geq \gamma_v$. As $p_v \wedge q_v$ is minimal and central in \mathcal{N}_0 , we see that

$$\mathcal{N}_1 = M_3 \otimes L(\mathbb{Z}) \oplus \begin{matrix} p_v^I \wedge q_v^I \\ M_2 \\ 3(\gamma_v - \gamma_w) \\ 2\gamma_w - \gamma_v \end{matrix}.$$

By [Dyk93], the projections p_v^I and q_v^I are equivalent in \mathcal{N}_0 , so it follows that p_w is equivalent to q_v^I in \mathcal{N}_1 . Therefore by Lemma 2.5,

$$p_w \mathcal{N}_2 p_w = p_w \mathcal{N}_1 p_w * L(\mathbb{Z}) = \left(L(\mathbb{Z}) \oplus \begin{matrix} \mathbb{C} \\ \gamma_w - \gamma_w \\ \gamma_w \end{matrix} \right) * L(\mathbb{Z}),$$

which is an interpolated free group factor $L(\mathbb{F}_t)$ [Dyk93]. As the central support of p_w in \mathcal{N}_2 is 1, it follows that \mathcal{N}_2 is also an interpolated free group factor $L(\mathbb{F}_{t_1})$. To

finish up this case, we write

$$\begin{aligned}\mathcal{N}_2 \subset \mathcal{N}_3 &= \left(M_2 \otimes L(\mathbb{Z}) \oplus_{2\gamma_w} \mathbb{C}_{\gamma_v - \gamma_w} \right) *_D \left(\begin{smallmatrix} p_w \\ M_2 \oplus \mathbb{C} \\ \gamma_w \quad \gamma_v - \gamma_w \end{smallmatrix} \right) \text{ and} \\ \mathcal{N}_3 \subset \mathcal{M}(\Gamma) &= \left(M_2 \otimes L(\mathbb{Z}) \oplus_{2\gamma_w} \mathbb{C}_{\gamma_v - \gamma_w} \right) *_D \left(M_2 \otimes L(\mathbb{Z}) \oplus_{2\gamma_w} \mathbb{C}_{\gamma_v - \gamma_w} \right),\end{aligned}$$

and use Lemma 2.4 twice, as well as the amplification formula for interpolated free group factors to obtain that $\mathcal{M}(\Gamma)$ is an interpolated free group factor.

Case 1b: The case $\gamma_v > 2\gamma_w$ for \mathcal{N}_0 is treated exactly the same as the first with only the caveat that the central support of p_w in \mathcal{N}_1 is a projection of trace $3\gamma_w$, so \mathcal{N}_1 , and thus $\mathcal{N}_2, \mathcal{N}_3$, and $\mathcal{M}(\Gamma)$, have a minimal central projection of trace $\gamma_v - 2\gamma_w$.

Case 2: Next we consider the case where Γ has 3 vertices v_1, v_2 , and v_3 with weights γ_1, γ_2 , and γ_3 respectively, where v_2 is connected to v_1 by e_1 and to v_3 by e_2 . There are two sub-cases to consider. The first is when $\gamma_2 \geq \gamma_1 \geq \gamma_3$, and the second is when $\gamma_1 > \gamma_2$ and $\gamma_1 \geq \gamma_3$.

Case 2a: We examine the following sequence of inclusions:

$$\begin{aligned}\mathcal{N}_0 &= \left(\begin{smallmatrix} p_{v_1} \\ \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ \gamma_1 \quad \gamma_1 \quad \gamma_2 - \gamma_1 \quad \gamma_3 \end{smallmatrix} \right) *_D \left(\begin{smallmatrix} p_{v_1} & q_2^I & q_2^{II} & p_{v_3} \\ \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \\ \gamma_1 \quad \gamma_2 - \gamma_3 \quad \gamma_3 \quad \gamma_3 \end{smallmatrix} \right) \\ &\cap \\ \mathcal{N}_1 &= \left(\begin{smallmatrix} p_{v_1}, p_2^I & p_2^{II} & p_{v_3} \\ M_2 \oplus \mathbb{C} \oplus \mathbb{C} \\ \gamma_1 \quad \gamma_2 - \gamma_1 \quad \gamma_3 \end{smallmatrix} \right) *_D \left(\begin{smallmatrix} p_{v_1} & q_2^I & q_2^{II}, p_{v_3} \\ \mathbb{C} \oplus \mathbb{C} \oplus M_2 \\ \gamma_1 \quad \gamma_2 - \gamma_3 \quad \gamma_3 \end{smallmatrix} \right) \\ &\cap \\ \mathcal{N}_2 &= \left(M_2 \otimes L(\mathbb{Z}) \oplus_{2\gamma_1} \begin{smallmatrix} p_2^{II} \\ \mathbb{C} \oplus \mathbb{C} \\ \gamma_2 - \gamma_1 \quad \gamma_3 \end{smallmatrix} \right) *_D \left(\begin{smallmatrix} p_{v_1} & q_2^I & q_2^{II}, p_{v_3} \\ \mathbb{C} \oplus \mathbb{C} \oplus M_2 \\ \gamma_1 \quad \gamma_2 - \gamma_3 \quad \gamma_3 \end{smallmatrix} \right) \\ &\cap \\ \mathcal{M}(\Gamma) &= \left(M_2 \otimes L(\mathbb{Z}) \oplus_{\gamma_1} \begin{smallmatrix} p_2^{II} \\ \mathbb{C} \oplus \mathbb{C} \\ \gamma_2 - \gamma_1 \quad \gamma_3 \end{smallmatrix} \right) *_D \left(\begin{smallmatrix} p_{v_1} & q_2^I & M_2 \otimes L(\mathbb{Z}) \\ \mathbb{C} \oplus \mathbb{C} \oplus \\ \gamma_1 \quad \gamma_2 - \gamma_3 \quad 2\gamma_3 \end{smallmatrix} \right),\end{aligned}$$

where p_{v_2} decomposes as $1 \otimes e_{22} \oplus 1 \oplus 0$ in \mathcal{A}_{e_1} and $0 \oplus 1 \oplus 1 \otimes e_{1,1}$ in \mathcal{A}_{e_2} . We set p_2^I and p_2^{II} as the summands of p_{v_2} supported in the diffuse and atomic parts of \mathcal{A}_{e_1} respectively and q_2^I and q_2^{II} as the summands of p_{v_2} supported in the atomic and diffuse parts of \mathcal{A}_{e_2} respectively. As above,

$$p_{v_2} \mathcal{N}_0 p_{v_2} = \begin{smallmatrix} p_2^I & p_2^{II} & q_2^I & q_2^{II} \\ \mathbb{C} \oplus \mathbb{C} & * & \mathbb{C} \oplus \mathbb{C} \\ \frac{\gamma_1}{\gamma_2} & \frac{\gamma_2 - \gamma_1}{\gamma_2} & \frac{\gamma_2 - \gamma_3}{\gamma_2} & \frac{\gamma_3}{\gamma_2} \end{smallmatrix} = \begin{cases} M_2 \otimes L(\mathbb{Z}) \oplus_{2\frac{\gamma_2 - \gamma_1}{\gamma_2}} \begin{smallmatrix} p_2^I \wedge q_2^I & p_2^I \wedge q_2^{II} \\ \mathbb{C} \oplus \mathbb{C} \\ \frac{\gamma_1 - \gamma_3}{\gamma_2} & \frac{\gamma_1 - \gamma_2 + \gamma_3}{\gamma_2} \end{smallmatrix} & \text{if } \gamma_2 \leq \gamma_1 + \gamma_3 \\ M_2 \otimes L(\mathbb{Z}) \oplus_{2\frac{\gamma_3}{\gamma_2}} \begin{smallmatrix} p_2^I \wedge q_2^I & p_2^{II} \wedge q_2^I \\ \mathbb{C} \oplus \mathbb{C} \\ \frac{\gamma_1 - \gamma_3}{\gamma_2} & \frac{\gamma_2 - \gamma_1 - \gamma_3}{\gamma_2} \end{smallmatrix} & \text{if } \gamma_2 > \gamma_1 + \gamma_3 \end{cases}.$$

Case 2a(i): Assume $\gamma_2 \leq \gamma_1 + \gamma_3$. Since the two new matrix units in \mathcal{N}_1 introduce equivalences between p_{v_1} and p_2^I and between q_2^{II} and p_{v_3} respectively, we see that \mathcal{N}_1 has the same number of summands as $p_{v_2}\mathcal{N}_0p_{v_2}$, but with suitable amplifications. Explicitly, we find that

$$\mathcal{N}_1 = M_4 \otimes L(\mathbb{Z}) \oplus \underset{4(\gamma_2 - \gamma_1)}{M_2} \oplus \underset{\gamma_1 - \gamma_3}{p_2^I \wedge q_2^I} \oplus \underset{\gamma_1 + \gamma_3 - \gamma_2}{p_2^I \wedge q_2^{II}} M_3$$

where the central support of p_{v_1} is the identity. By applying Lemma 2.4 and applying the same reasoning as case 1, we see that \mathcal{N}_2 is an interpolated free group factor. Applying Lemma 2.4 again shows that $\mathcal{M}(\Gamma)$ is an interpolated free group factor.

Case 2a(ii): Assume $\gamma_2 > \gamma_1 + \gamma_3$. This case is treated in the same way as above except that in \mathcal{N}_1 , $q_2^I \wedge p_2^{II}$ with trace $\gamma_2 - \gamma_3 - \gamma_1$ is minimal and central, so it is minimal and central in $\mathcal{M}(\Gamma)$.

Case 2b: Now let γ_1 be the largest weight. First assume $\gamma_3 \geq \gamma_2$. We consider the algebra

$$\mathcal{N}_1 = \left(\underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{\gamma_2}{p_{v_2}} M_2 \oplus \underset{\gamma_3}{p_{v_3}} \right) *_D \left(\underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2}{p_{v_2}} M_2 \oplus \underset{\gamma_3 - \gamma_2}{p_3^I} \right),$$

where the projections orthogonal to p_{v_2} in each copy of M_2 sit under p_i and $p_i^I \leq p_{v_i}$ for $i = 1$ or 3 . It follows that $\mathcal{N}_1 = \underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{\gamma_2}{p_{v_2}} M_3 \oplus \underset{\gamma_3 - \gamma_2}{p_3^I}$, so tensoring each copy of M_2 with $L(\mathbb{Z})$ and using the standard arguments as above show that

$$\mathcal{M}(\Gamma) = \underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{3\gamma_2}{L(\mathbb{F}_t)} \oplus \underset{\gamma_3 - \gamma_2}{p_3^I}.$$

Finally, if $\gamma_2 > \gamma_3$ then we consider

$$\mathcal{N}_1 = \left(\underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{\gamma_2}{p_{v_2}} M_2 \oplus \underset{\gamma_3}{p_{v_3}} \right) *_D \left(\underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2 - \gamma_3}{p_2^I} \oplus \underset{\gamma_3}{p_{v_3}} \right) = \underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{\gamma_3}{M_3} \oplus \underset{\gamma_2 - \gamma_3}{M_2},$$

where the central support of p_{v_2} is $1 - p_1^I$. Therefore, tensoring each copy of M_2 with $L(\mathbb{Z})$ gives $\mathcal{M}(\Gamma) = \underset{\gamma_1 - \gamma_2}{\mathbb{C}} \oplus \underset{2\gamma_2 + \gamma_3}{L(\mathbb{F}_t)}$ as desired. \square

We now inductively assume that for some Γ , $\mathcal{M}(\Gamma)$ has the form as described in Theorem 3.4.

Lemma 3.6. *Suppose Γ' is a graph obtained from Γ by adding an edge e connecting two vertices v and w of Γ (so that in particular Γ and Γ' have the same underlying set of vertices with the same weighting). Assume that*

$$\mathcal{M}(\Gamma) = L(\mathbb{F}_{t_\Gamma}^{p^\Gamma}) \oplus \bigoplus_{v \in B(\Gamma)} \mathbb{C}_{\gamma_v - \alpha_v^\Gamma}^{r_v^\Gamma}$$

as in Theorem 3.4. Then

$$\mathcal{M}(\Gamma') = L(\mathbb{F}_{t_{\Gamma'}})^{p^{\Gamma'}} \oplus \bigoplus_{v \in B(\Gamma')} \mathbb{C}^{r_v^{\Gamma'} - \alpha_v^{\Gamma'}}_{\gamma_v - \alpha_v^{\Gamma'}}$$

where $p^\Gamma \leq p^{\Gamma'}$, $r_v^{\Gamma'} \leq r_v^\Gamma$, and $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p_\Gamma \mathcal{M}(\Gamma') p_\Gamma$.

Proof. We use the convention that if the term \mathbb{C}^α appears where $\alpha \leq 0$ then this term is identically zero. All parts of the proof below are valid if this modification is made.

Set $D = \ell^\infty(\Gamma') = \ell^\infty(\Gamma)$ and let the new edge e connect v to w with $\gamma_v \geq \gamma_w$. We examine the following sequence of inclusions:

$$\begin{aligned} \mathcal{M}(\Gamma) &\subset \mathcal{N}_1 = \mathcal{M}(\Gamma) \underset{D}{*} \left(\mathbb{C}^{p_w}_{\gamma_w} \oplus \left(\bigoplus_{k=1}^n \mathbb{C}^{p_{w,k}}_{\gamma_w/n} \oplus \mathbb{C}^{p_v^I}_{\gamma_v - \gamma_w} \right) \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) \\ &\cap \\ \mathcal{N}_2 &= \mathcal{M}(\Gamma) \underset{D}{*} \left(\left(\bigoplus_{k=1}^n \mathbb{C}^{p_{w,k}}_{\gamma_w/n} \oplus \left(\bigoplus_{k=1}^n \mathbb{C}^{p_{v,k}}_{\gamma_w/n} \oplus \mathbb{C}^{p_v^I}_{\gamma_v - \gamma_w} \right) \right) \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) \\ &\cap \\ \mathcal{N}_3 &= \mathcal{M}(\Gamma) \underset{D}{*} \left(\left(\bigoplus_{k=1}^n M_2^{p_{w,k}, p_{v,k}}_{\gamma_w/n} \oplus \mathbb{C}^{p_v^I}_{\gamma_v - \gamma_w} \right) \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) \\ &\cap \\ \mathcal{M}(\Gamma') &= \mathcal{M}(\Gamma) \underset{D}{*} \left(L(\mathbb{Z}) \otimes M_2 \oplus \mathbb{C}^{p_v^I}_{\gamma_v - \gamma_w} \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right). \end{aligned}$$

The projections $p_{w,k}$ are an orthogonal family with trace γ_w/n in \mathcal{A}_e whose sum is p_w . The projection p_v decomposes as $\sum_{k=1}^n p_{v,k} + p_v^I$ with p_v^I supported in the atomic part of \mathcal{A}_e and the $p_{v,k}$ are an orthogonal family of projections with trace γ_w/n supported in the diffuse part of \mathcal{A}_e . The positive integer n is chosen large enough such that $\frac{1}{n} + \frac{\gamma_w - \alpha_w^\Gamma}{\gamma_w} < 1$ and $\frac{\gamma_w}{n\gamma_v} + \frac{\gamma_v - \alpha_v^\Gamma}{\gamma_v} < 1$. From the induction hypothesis,

$$p_v \mathcal{M}(\Gamma) p_v = L(\mathbb{F}_{t_v})^{p_v^\Gamma} \oplus \mathbb{C}^{r_v^\Gamma}_{\frac{\gamma_v - \alpha_v^\Gamma}{\gamma_v}}, \text{ and } p_w \mathcal{M}(\Gamma) p_w = L(\mathbb{F}_{t_w})^{p_w^\Gamma} \oplus \mathbb{C}^{r_w^\Gamma}_{\frac{\gamma_w - \alpha_w^\Gamma}{\gamma_w}},$$

with $p_u^\Gamma = p^\Gamma p_u$ for any vertex u . From Lemma 2.4,

$$p_v \mathcal{N}_1 p_v = \left(L(\mathbb{F}_{t_v})^{p_v^\Gamma} \oplus \mathbb{C}^{r_v^\Gamma}_{\frac{\gamma_v - \alpha_v^\Gamma}{\gamma_v}} \right) * \left(\bigoplus_{k=1}^n \mathbb{C}^{p_{v,k}}_{\frac{\gamma_w}{n\gamma_v}} \oplus \mathbb{C}^{p_v^I}_{\frac{\gamma_v - \gamma_w}{\gamma_v}} \right) = L(\mathbb{F}_{t_{v,1}})^{p_v^\Gamma \wedge r_v^\Gamma} \oplus \mathbb{C}^{p_v^\Gamma \wedge r_v^\Gamma}_{\frac{\gamma_v - \alpha_v^{\Gamma'}}{\gamma_v}}.$$

Lemma 2.4 applied to the inclusion

$$\left(\begin{smallmatrix} p_v^\Gamma & r_v^\Gamma \\ \mathbb{C} & \mathbb{C} \\ \gamma_v - \alpha_v & \gamma_v \end{smallmatrix} \right) * \left(\bigoplus_{k=1}^n \begin{smallmatrix} p_{v,k} & p_v^I \\ \mathbb{C} & \mathbb{C} \\ \gamma_w & \gamma_w - \gamma_v \end{smallmatrix} \right) \rightarrow \left(L(\mathbb{F}_{t_v}) \oplus \begin{smallmatrix} r_v^\Gamma \\ \mathbb{C} \\ \gamma_v \end{smallmatrix} \right) * \left(\bigoplus_{k=1}^n \begin{smallmatrix} p_{v,k} & p_v^I \\ \mathbb{C} & \mathbb{C} \\ \gamma_w & \gamma_w - \gamma_v \end{smallmatrix} \right),$$

shows that the inclusion $L(\mathbb{F}_{t_v}) = p_v^\Gamma \mathcal{M}(\Gamma) p_v^\Gamma \rightarrow p_v^\Gamma \mathcal{N}_1 p_v^\Gamma$ is equivalent to the canonical inclusion

$$L(\mathbb{F}_{t_v}) \rightarrow L(\mathbb{F}_{t_v}) * p_v^\Gamma \left[\left(\begin{smallmatrix} p_v' & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \\ \gamma_v - \alpha_v & \gamma_v \end{smallmatrix} \right) * \left(\bigoplus_{k=1}^n \begin{smallmatrix} p_{v,k} & p_v^I \\ \mathbb{C} & \mathbb{C} \\ \gamma_w & \gamma_w - \gamma_v \end{smallmatrix} \right) \right] p_v^\Gamma$$

so $p_v^\Gamma \mathcal{M}(\Gamma) p_v^\Gamma \xrightarrow{s.e.} p_v^\Gamma \mathcal{N}_1 p_v^\Gamma$. From Remark 2.3, $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_1 p^\Gamma$ as well.

By Lemma 2.4 we have

$$p_w \mathcal{N}_2 p_w = L(\mathbb{Z}/n\mathbb{Z}) * p_w \mathcal{N}_1 p_w = L(\mathbb{Z}/n\mathbb{Z}) * \left(L(\mathbb{F}_{t_w}) \oplus \begin{smallmatrix} r_w^{\mathcal{N}_1} \\ \mathbb{C} \\ \gamma_w \end{smallmatrix} \right),$$

where $p_w^{\mathcal{N}_1} = p_w p^{\mathcal{N}_1}$ with $p^{\mathcal{N}_1}$ the central support of p_Γ in \mathcal{N}_1 (note $p_w^{\mathcal{N}_1} \geq p_w^\Gamma$ so $r_w^{\mathcal{N}_1} \leq r_w^\Gamma$ which implies $\delta_w \leq \gamma_w - \alpha_w$). From these observations, it follows that $p_w \mathcal{N}_2 p_w$ is an interpolated free group factor (since n was chosen such that $\frac{\gamma_w}{n\gamma_v} + \frac{\gamma_w - \alpha_v}{\gamma_v} < 1$) and the arguments used in the inclusion $\mathcal{M}(\Gamma) \rightarrow \mathcal{N}_1$ imply $p_w^{\mathcal{N}_1} \mathcal{N}_1 p_w^{\mathcal{N}_1} \xrightarrow{s.e.} p_w^{\mathcal{N}_1} \mathcal{N}_2 p_w^{\mathcal{N}_1}$. Therefore $p_w^\Gamma \mathcal{N}_1 p_w^\Gamma \xrightarrow{s.e.} p_w^\Gamma \mathcal{N}_2 p_w^\Gamma$ so $p^\Gamma \mathcal{N}_1 p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_2 p^\Gamma$. Also, observe that since the projections $p_{v,k}$ and $p_{w,k}$ lie in the interpolated free group factor summand of \mathcal{N}_2 , they are equivalent in \mathcal{N}_2 . We now define algebras $\mathcal{N}_{2,j}$ for $j = 0, \dots, n$ so that

$$\mathcal{N}_2 = \mathcal{N}_{2,0} \subset \mathcal{N}_{2,1} \subset \mathcal{N}_{2,2} \subset \dots \subset \mathcal{N}_{2,n} = \mathcal{N}_2 \text{ where}$$

$$\mathcal{N}_{2,j} = \left(\bigoplus_{k=j+1}^n \begin{smallmatrix} p_{w,k} & p_{v,k} \\ \mathbb{C} & \mathbb{C} \\ \gamma_w/n & \gamma_w/n \end{smallmatrix} \oplus \bigoplus_{k=1}^j \begin{smallmatrix} p_{w,k} & p_{v,k} \\ M_2 & M_2 \\ \gamma_w/n & \gamma_w/n \end{smallmatrix} \oplus \bigoplus_{k=j+1}^n \begin{smallmatrix} p_{v,k} & p_v^I \\ \mathbb{C} & \mathbb{C} \\ \gamma_w/n & \gamma_w - \gamma_v \end{smallmatrix} \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right)_D * \mathcal{M}(\Gamma).$$

Let $p^{\mathcal{N}_2}$ be the central support of p_Γ in \mathcal{N}_2 . Applying Lemma 2.5 to the inclusion

$$p_{w,j+1} \mathcal{N}_{2,j} p_{w,j+1} \rightarrow p_{w,j+1} \mathcal{N}_{2,j+1} p_{w,j+1} = p_{w,j+1} \mathcal{N}_{2,j} p_{w,j+1} * L(\mathbb{Z})$$

shows that this inclusion is a standard embedding, so it follows from Remark 2.3 that $p^{\mathcal{N}_2} \mathcal{N}_{2,j} p^{\mathcal{N}_2} \xrightarrow{s.e.} p^{\mathcal{N}_2} \mathcal{N}_{2,j+1} p^{\mathcal{N}_2}$, implying $p^\Gamma \mathcal{N}_{2,j} p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_{2,j+1} p^\Gamma$ for all j . Inductively,

$$\mathcal{N}_3 = \left(L(\mathbb{F}_{t_3}) \oplus \begin{smallmatrix} p^{\mathcal{N}_2} & p_v^I \wedge r_v^\Gamma \\ \mathbb{C} & \mathbb{C} \\ \gamma_v - \alpha_v^{\Gamma'} & \gamma_v - \alpha_v^{\Gamma'} \end{smallmatrix} \oplus \bigoplus_{u \in L(\Gamma) \setminus \{v, w\}} \begin{smallmatrix} r_u^\Gamma \\ \mathbb{C} \\ \gamma_u - \alpha_u^{\Gamma'} \end{smallmatrix} \right)$$

and $p^\Gamma \mathcal{N}_2 p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_3 p^\Gamma$. To finish, we look at the sequence of algebras

$$\mathcal{N}_3 = \mathcal{N}_{3,0} \subset \mathcal{N}_{3,1} \subset \dots \subset \mathcal{N}_{3,n} = \mathcal{M}(\Gamma') \text{ where}$$

$$\mathcal{N}_{3,j} = \left(\bigoplus_{k=1}^j M_2 \otimes L(\mathbb{Z}) \oplus \bigoplus_{k=j+1}^n M_2 \oplus \bigoplus_{\gamma_v - \gamma_w} \mathbb{C} \oplus \ell^\infty(\Gamma \setminus \{v, w\}) \right) *_D \mathcal{M}(\Gamma).$$

Lemma 2.4 implies that the inclusion

$$p_{w,j+1} \mathcal{N}_{3,j} p_{w,j+1} \rightarrow p_{w,j+1} \mathcal{N}_{3,j+1} p_{w,j+1} = p_{w,j+1} \mathcal{N}_{3,j} p_{w,j+1} * L(\mathbb{Z})$$

is a standard embedding, so by Remark 2.3, $p^{\mathcal{N}_2} \mathcal{N}_{3,j} p^{\mathcal{N}_2} \xrightarrow{s.e.} p^{\mathcal{N}_2} \mathcal{N}_{3,j+1} p^{\mathcal{N}_2}$ and thus $p^\Gamma \mathcal{N}_{3,j} p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_{3,j+1} p^\Gamma$. Therefore the inclusion $p^\Gamma \mathcal{N}_3 p^\Gamma \rightarrow p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$ is standard since it is a composite of standard embeddings. This implies $\mathcal{M}(\Gamma')$ has the desired formula and $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$. \square

We again assume that $\mathcal{M}(\Gamma)$ is in the form of Theorem 3.4.

Lemma 3.7. *Let Γ' be a weighted graph obtained from Γ by adding a vertex v and an edge e connecting v to $w \in V(\Gamma)$ with weighting γ_v , and assume $\mathcal{M}(\Gamma) = L(\mathbb{F}_{t_\Gamma}^\Gamma) \oplus \bigoplus_{v \in B(\Gamma) \gamma_v - \alpha_v^\Gamma} \mathbb{C}^{r_v^\Gamma}$ with notation as in Theorem 3.4. Then*

$$\mathcal{M}(\Gamma') = L(\mathbb{F}_{t_{\Gamma'}}^{p^{\Gamma'}}) \oplus \bigoplus_{u \in B(\Gamma') \gamma_u - \alpha_u^{\Gamma'}} \mathbb{C}^{r_u^{\Gamma'}}$$

where $p^\Gamma \leq p^{\Gamma'}$, $r_u^\Gamma \leq r_u^{\Gamma'}$ for all u , and $p_\Gamma \mathcal{M}(\Gamma) p_\Gamma \xrightarrow{s.e.} p_\Gamma \mathcal{M}(\Gamma') p_\Gamma$.

Notice that the natural inclusion $\mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma')$ is not unital, but the compressed inclusion $p_\Gamma \mathcal{M}(\Gamma) p_\Gamma \rightarrow p_\Gamma \mathcal{M}(\Gamma') p_\Gamma$ is.

Proof. Just as in the proof of Lemma 3.6, if the term \mathbb{C}_α^p appears where $\alpha \leq 0$ then this term is identically zero.

Set $D = \ell^\infty(\Gamma')$. We rescale all of the weights on Γ such that all of the weights on Γ' sum to 1. We have 2 cases: when $\gamma_v > \gamma_w$ and when $\gamma_w \geq \gamma_v$.

Case 1, $\gamma_v > \gamma_w$: We look at the following sequence of inclusions:

$$\begin{aligned} \mathcal{M}(\Gamma) \oplus \mathbb{C}_{\gamma_v}^{p_v} &\subset \mathcal{N}_1 = \left(\mathcal{M}(\Gamma) \oplus \mathbb{C}_{\gamma_v}^{p_v} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n \mathbb{C}_{\gamma_w/n}^{p_{w,k}} \oplus \left(\bigoplus_{k=1}^n \mathbb{C}_{\gamma_w/n}^{p_{v,k}} \oplus \mathbb{C}_{\gamma_v - \gamma_w}^{p_v^I} \right) \right) \\ &\cap \\ \mathcal{N}_2 &= \left(\mathcal{M}(\Gamma) \oplus \mathbb{C}_{\gamma_v}^{p_v} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n M_2 \oplus \mathbb{C}_{\gamma_v - \gamma_w}^{p_v^I} \right) \\ &\cap \\ \mathcal{M}(\Gamma') &= \left(\mathcal{M}(\Gamma) \oplus \mathbb{C}_{\gamma_v}^{p_v} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus L(\mathbb{Z}) \otimes_{2\gamma_w} M_2 \oplus \mathbb{C}_{\gamma_v - \gamma_w}^{p_v^I} \right). \end{aligned}$$

The projections $p_{w,k}$ are an orthogonal family with trace γ_w/n in \mathcal{A}_e whose sum is p_w . In \mathcal{A}_e , p_v decomposes as $\sum_{k=1}^n p_{v,k} + p_v^I$ with p_v^I supported in the atomic part of \mathcal{A}_e , and the $p_{v,k}$ are an orthogonal family of projections with trace γ_w/n supported in the diffuse part of \mathcal{A}_e . By the inductive hypothesis,

$$p_w \mathcal{M}(\Gamma) p_w = L(\mathbb{F}_{t_w}) \oplus \underset{\frac{\alpha_w}{\gamma_w}}{\mathbb{C}}^{p_w^\Gamma} \oplus \underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\mathbb{C}}^{r_w^\Gamma},$$

with $p_w^\Gamma = p_w p^\Gamma$. We choose n large enough such that $\frac{1}{n} + \frac{\gamma_w - \alpha_w}{\gamma_w} < 1$, i.e., so that $p_w \mathcal{M}(\Gamma) p_w * L(\mathbb{Z}/n\mathbb{Z})$ is an interpolated free group factor. From Lemma 2.4,

$$p_w \mathcal{N}_1 p_w = p_w \mathcal{M}(\Gamma) p_w * \left(\bigoplus_{k=1}^n \underset{1/n}{\mathbb{C}}^{p_{w,k}} \right) = \left(L(\mathbb{F}_{t_w}) \oplus \underset{\frac{\alpha_w}{\gamma_w}}{\mathbb{C}}^{p_w^\Gamma} \oplus \underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\mathbb{C}}^{r_w^\Gamma} \right) * \left(\bigoplus_{k=1}^n \underset{1/n}{\mathbb{C}}^{p_{w,k}} \right),$$

so it is an interpolated free group factor, and applying Lemma 2.4 again, we see that

$$p_w^\Gamma \mathcal{N}_1 p_w^\Gamma = p_w^\Gamma \mathcal{M}(\Gamma) p_w^\Gamma * p_w^\Gamma \left[\left(\underset{\frac{\alpha_w}{\gamma_w}}{\mathbb{C}}^{p_w^\Gamma} \oplus \underset{\frac{\gamma_w - \alpha_w}{\gamma_w}}{\mathbb{C}}^{r_w^\Gamma} \right) * \left(\bigoplus_{k=1}^n \underset{1/n}{\mathbb{C}}^{p_{w,k}} \right) \right] p_w^\Gamma$$

with the inclusion $p_w^\Gamma \mathcal{M}(\Gamma) p_w^\Gamma \rightarrow p_w^\Gamma \mathcal{N}_1 p_w^\Gamma$ the canonical one. Therefore $p_w^\Gamma \mathcal{M}(\Gamma) p_w^\Gamma \xrightarrow{s.e.} p_w^\Gamma \mathcal{N}_1 p_w^\Gamma$, so it follows that $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{N}_1 p^\Gamma$ as well. It is clear that p_v^I will be a minimal central projection in \mathcal{N}_2 , and since the projections $p_{v,k}$ lie under the minimal projection $p_v \in \mathcal{M}(\Gamma) \oplus \underset{\gamma_v}{\mathbb{C}}^{p_v^I}$, it follows that

$$\mathcal{N}_2 = L(\mathbb{F}_{t_2}) \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}}^{p_v^I} \oplus \bigoplus_{u \in B(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u^\Gamma}{\mathbb{C}}^{r_u^\Gamma},$$

where $L(\mathbb{F}_{t_2})$ is an amplification of $p^\Gamma \mathcal{N}_1 p^\Gamma$. Hence $p^\Gamma \mathcal{N}_1 p^\Gamma = p^\Gamma \mathcal{N}_2 p^\Gamma$. As a final step, we tensor each copy of M_2 with $L(\mathbb{Z})$ to obtain $\mathcal{M}(\Gamma')$ and apply Lemma 2.4 and Remark 2.3 n times as in the proof of Lemma 3.6 to conclude that

$$\mathcal{M}(\Gamma') = L(\mathbb{F}_{t_3}) \oplus \underset{\gamma_v - \gamma_w}{\mathbb{C}}^{p_v^I} \oplus \bigoplus_{u \in B(\Gamma) \setminus \{w\}} \underset{\gamma_u - \alpha_u^{\Gamma'}}{\mathbb{C}}^{r_u^\Gamma}$$

and $p^\Gamma \mathcal{N}_2 p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$. Therefore $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \xrightarrow{s.e.} p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$ and $\mathcal{M}(\Gamma')$ has the desired form.

Case 2, $\gamma_w \geq \gamma_v$: We look at a sequence of inclusions similar to those in the

previous case:

$$\begin{aligned}
\mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C} &\subset \mathcal{N}_1 = \left(\mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \left(\bigoplus_{k=1}^n \mathbb{C}_{\gamma_v/n}^{p_{w,k}} \oplus \mathbb{C}_{\gamma_w-\gamma_v}^{p_w^I} \right) \oplus \bigoplus_{k=1}^n \mathbb{C}_{\gamma_v/n}^{p_{v,k}} \right) \\
&\cap \\
\mathcal{N}_2 &= \left(\mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n \mathbb{C}_{\gamma_v/n}^{p_{w,k}, p_{v,k}} \oplus \mathbb{C}_{\gamma_w-\gamma_v}^{p_w^I} \right) \\
&\cap \\
\mathcal{M}(\Gamma') &= \left(\mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus L(\mathbb{Z}) \otimes_{2\gamma_v} M_2 \oplus \mathbb{C}_{\gamma_w-\gamma_v}^{p_w^I} \right).
\end{aligned}$$

The projections $p_{v,k}$ are an orthogonal family with trace γ_v/n in \mathcal{A}_e whose sum is p_v . In \mathcal{A}_e , p_w decomposes as $\sum_{k=1}^n p_{w,k} + p_w^I$ where p_w^I is supported in the atomic part of \mathcal{A}_e , and the $p_{w,k}$ are an orthogonal family of projections with trace γ_v/n supported in the diffuse part of \mathcal{A}_e . We choose n large enough so that $\frac{\gamma_w - \alpha_w}{\gamma_v} + \frac{\gamma_v}{n\gamma_w} < 1$.

Observe by the condition on n that $p_w \mathcal{N}_1 p_w = L(\mathbb{F}_{t'_1}^{p_w^{\mathcal{N}_1}}) \oplus \mathbb{C}_{\gamma_w - \alpha_w - \gamma_v}$ where the copy of \mathbb{C} is orthogonal to each $p_{w,k}$. Therefore as in the proof of Lemma 3.6 $p_\Gamma \mathcal{M}(\Gamma) p_\Gamma \xrightarrow{s.e.} p_\Gamma \mathcal{N}_1 p_\Gamma$. We next look at

$$\mathcal{N}_1 \subset \mathcal{N}_2 = \left(\mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C} \right) *_D \left(\ell^\infty(\Gamma' \setminus \{v, w\}) \oplus \bigoplus_{k=1}^n \mathbb{C}_{\gamma_v/n}^{p_{w,k}, p_{v,k}} \oplus \mathbb{C}_{\gamma_w-\gamma_v}^{p_w^I} \right).$$

Since the $p_{v,k}$ lie under the minimal central projection $p_v \in \mathcal{M}(\Gamma) \oplus_{\gamma_v}^{p_v} \mathbb{C}$, the arguments above imply

$$\mathcal{N}_2 = L(\mathbb{F}_{t_2}) \oplus \mathbb{C}_{\gamma_w - \gamma_v - \alpha_w} \oplus \bigoplus_{u \in L(\Gamma) \setminus \{w\}} \mathbb{C}_{\gamma_u - \alpha_u}^{r_u^\Gamma}$$

and $p^\Gamma \mathcal{N}_1 p^\Gamma = p^\Gamma \mathcal{N}_2 p^\Gamma$. To finish, we tensor each copy of M_2 with $L(\mathbb{Z})$ and apply Lemma 2.4 and Remark 2.3 n times as in the end of the proof of Lemma 3.6 to obtain

$$\mathcal{M}(\Gamma') = L(\mathbb{F}_{t_3}) \oplus \mathbb{C}_{\gamma_w - \gamma_v - \alpha_w} \oplus \bigoplus_{u \in L(\Gamma) \setminus \{w\}} \mathbb{C}_{\gamma_u - \alpha_u}^{r_u^\Gamma}$$

with the inclusion $p^\Gamma \mathcal{M}(\Gamma) p^\Gamma \rightarrow p^\Gamma \mathcal{M}(\Gamma') p^\Gamma$ standard. \square

Proof of Theorem 3.4. Note that if Γ' and Γ are connected, loopless, finite graphs, then Γ' can be constructed from Γ by considering the steps in Lemmas 3.6 and 3.7. Therefore, we can deduce Theorem 3.4 by observing that the composite of standard embeddings is a standard embedding and that standard embeddings are preserved by cut-downs by projections. \square

4 The GJS construction in infinite depth

Recall that the vertices on a principal graph for $A_0 \subset A_1$ represent isomorphism classes of irreducible $A_0 - A_0$ and $A_0 - A_1$ subbimodules of tensor products of $X =_{A_0} L^2(A_1)_{A_1}$ and its dual, $X^* =_{A_1} L^2(A_1)_{A_0}$. Assume Γ is the principal graph for an infinite-depth subfactor. If $*$ is the depth-0 vertex of Γ , then the factor A_0 as in the introduction is isomorphic to $p_* \mathcal{M}(\Gamma) p_*$. $\mathcal{M}(\Gamma)$ is now a semifinite algebra where the weighting γ on $\ell^\infty(\Gamma)$ corresponds to the bimodule dimension obtained by identifying each vertex with an irreducible bimodule as above. Under this identification, $\gamma_* = 1$ and $\delta \cdot \gamma_v = \sum_{w \sim v} n_{v,w} \gamma_w$ where $\delta = [A_1 : A_0]^{1/2}$. To circumvent the difficulty of dealing with a semifinite algebra, we realize that A_0 is an inductive limit of the algebras $p_* \mathcal{M}(\Gamma_k) p_*$ where Γ_k is Γ truncated at depth k . To aid our computation of the isomorphism class of A_0 , we have the following lemma, whose proof is a routine calculation and is identical to that in [GJS11].

Lemma 4.1. *The free dimension of $\mathcal{M}(\Gamma_k)$ is*

$$1 + \frac{1}{\text{Tr}(F_k)^2} \left(- \sum_{v \in \Gamma_k} \gamma_v^2 + \sum_{v \in \Gamma_k} \sum_{w \sim v} n_{v,w} \gamma_v \gamma_w \right)$$

where $w \sim v$ means w is connected to v in Γ_k , $F_k = \sum_{u \in \Gamma_k} p_u$, and Tr is the trace on the semifinite algebra $\mathcal{M}(\Gamma)$.

Theorem 4.2. *Let \mathcal{P} be an infinite depth subfactor planar algebra. Then the factor A_0 in the construction of [GJS10] is isomorphic to $L(\mathbb{F}_\infty)$.*

Proof. For a given k , we write

$$\mathcal{M}(\Gamma_k) = \begin{matrix} p_k \\ L(\mathbb{F}_{t_k}) \\ \sum_{w \notin B(\Gamma_k)} \gamma_w + \sum_{v \in B(\Gamma_k)} \alpha_v^{\Gamma_k} \end{matrix} \oplus \bigoplus_{v \in B(\Gamma_k)} \begin{matrix} \Gamma_k \\ p_v^{\Gamma_k} \\ \mathbb{C} \\ \gamma_v - \alpha_v^{\Gamma_k} \end{matrix}.$$

The free dimension of this algebra is

$$1 + (t_k - 1) \left(\frac{\sum_{w \notin B(\Gamma_k)} \gamma_w + \sum_{v \in B(\Gamma_k)} \gamma_v}{\text{Tr}(F_k)} \right)^2 - \frac{\sum_{v \in B(\Gamma_k)} (\gamma_v - \alpha_v^{\Gamma_k})^2}{\text{Tr}(F_k)^2},$$

so by Lemma 4.1, we have the equation

$$(t_k - 1) \left(\sum_{w \notin B(\Gamma_k)} \gamma_w + \sum_{v \in B(\Gamma_k)} \alpha_v^{\Gamma_k} \right)^2 = \sum_{u \in \Gamma_k} \sum_{w \sim u} n_{u,w} \gamma_u \gamma_w - \sum_{u \in \Gamma_k} \gamma_u^2 + \sum_{v \in B(\Gamma_k)} (\gamma_v - \alpha_v^{\Gamma_k})^2.$$

Observe that in Γ_k , the vertices up to depth $k - 1$ are connected to all of their neighbors in Γ , so by the Perron-Frobenius condition and the fact that $\delta > 1$, none of these vertices are in $B(\Gamma_k)$. If we let $B'(\Gamma_k)$ be the vertices v at depth k with $\gamma_v \leq \sum_{w \sim v} n_{v,w} \gamma_w$, then the right hand side of the above equality becomes

$$\begin{aligned}
& (\delta - 1) \sum_{v \in \Gamma_{k-2}} \gamma_v^2 + \sum_{v \in \Gamma_{k-1} \setminus \Gamma_{k-2}} \gamma_v \left(-\gamma_v + \sum_{\substack{w \notin B(\Gamma_k) \\ w \sim v}} n_{v,w} \gamma_w + \sum_{\substack{w \in B(\Gamma_k) \\ w \sim v}} \alpha_w^{\Gamma_k} \right) \\
& + \sum_{v \in B'(\Gamma_k)} \gamma_v \left(-\gamma_v + \sum_{w \sim v} n_{v,w} \gamma_w \right)
\end{aligned}$$

where we have used $\alpha_v^{\Gamma_k} = \sum_{w \sim v} n_{v,w} \gamma_w$. This quantity majorizes $(\delta - 1) \sum_{v \in \Gamma_{k-2}} \gamma_v^2$. Since the bimodule dimensions of any irreducible subbimodule of $(X \otimes_{A_1} X^*)^{\otimes_{A_0} n}$ and $(X \otimes_{A_1} X^*)^{\otimes_{A_0} n} \otimes_{A_0} X$ are bounded below by 1, $\gamma_v \geq 1$ for all $v \in \Gamma$ so we conclude that

$$(t_k - 1) \left(\sum_{w \notin L(\Gamma_k)} \gamma_w + \sum_{v \in L(\Gamma_k)} \alpha_v^{\Gamma_k} \right)^2 \rightarrow \infty$$

as $k \rightarrow \infty$. From the amplification formula, $p_* \mathcal{M}(\Gamma_k) p_* = L(\mathbb{F}_{t'_k})$ where

$$t'_k = 1 + (t_k - 1) \left(\sum_{w \notin L(\Gamma_k)} \gamma_w + \sum_{v \in L(\Gamma_k)} \alpha_v^{\Gamma_k} \right)^2.$$

Hence $p_{\Gamma_k} \mathcal{M}(\Gamma_k) p_{\Gamma_k} \xrightarrow{s.e.} p_{\Gamma_k} \mathcal{M}(\Gamma_{k+1}) p_{\Gamma_k}$ so by Remark 2.3, $p_* \mathcal{M}(\Gamma_k) p_* \xrightarrow{s.e.} p_* \mathcal{M}(\Gamma_{k+1}) p_*$. As $p_* \mathcal{M}(\Gamma) p_*$ is the inductive limit of the $p_* \mathcal{M}(\Gamma_k) p_*$, it follows that $p_* \mathcal{M}(\Gamma) p_* = L(\mathbb{F}_t)$ where $t = \lim t'_k = \infty$. \square

Corollary 4.3. *The factors A_k are isomorphic to $L(\mathbb{F}_\infty)$.*

Proof. If k is even, then A_k is an amplification of A_0 so it follows for A_k . If k is odd, then A_k are cut-downs of $\mathcal{M}(\Gamma')$ with Γ' the dual principal graph of \mathcal{P} . Applying the same analysis as in Theorem 4.2 shows that $A_k \cong L(F_\infty)$. \square

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